REAL LATTICE[0]

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In the previous paper "Real Number Music [1]", I defined a new concept called the "Inverse-Euclidean Algorithm." I explained that it was a generator which generated a self-similar literal string reflecting the recursive structure of the continued fraction. You might be surprised that the Inverse-Euclidean Algorithm also generates musical scale and spectral distributions of tone. ("Literal string" means any one-dimensional arrangement of letters of the alphabet.)

In this paper, I cover the ways in detail which generate a literal string one to one correspondence with any real number and generalize the form.

FIBONACCI LATTICE

I knew about the FIBONACCI LATTICE in 1986. It was explained that it was a trace pattern along a line whose tangent was the Golden Mean (Fig. 1)[2].

There is another way to generate the same literal string.

Initial string is "AB"

Substitute "AB" for "A" and substitute "B" for "A".

Then we get a string "ABA".

Repeat the manipulation recursively [2].

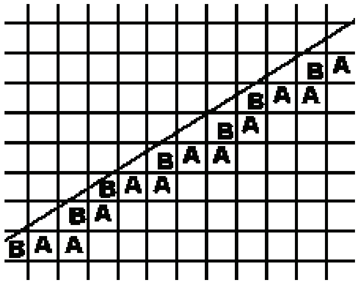

Fig. 1

This way is well known as the Fibonacci Rabbit and branching of tree.

Let the former way be called the "Tangent Method", and let the latter way be called the "Substitution Method".

I used both methods depending on my purpose but I had not solved why these two ways generated the same string. Actually, I found some other ways to generate the same strings as follows.

Phyllotaxy Method: dividing a circle or line segment by the golden mean (Fig. 2, 3). Most plants use this method.

Torus Method: a locus of a point which move diagonally on the square plane like Cathode Ray Tube (*f.e.* Computer Monitor Display) which is the same topology as the torus (Fig. 4).

Double Rings Method: the string is generated by sliding and transcripting two rings like DNA (Fig. 5).

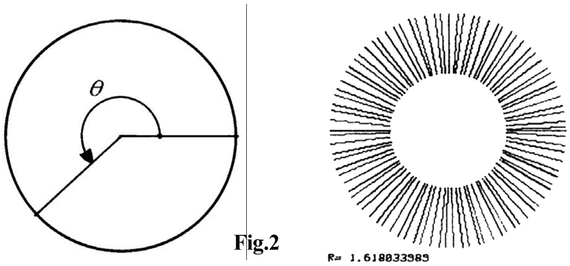

Fig.2

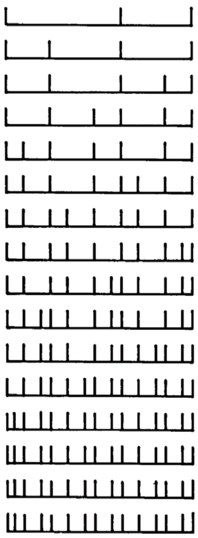

Fig. 3

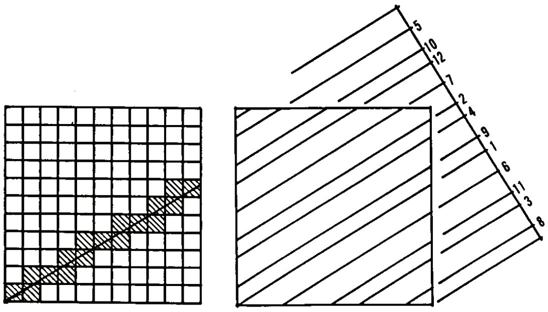

Fig. 4

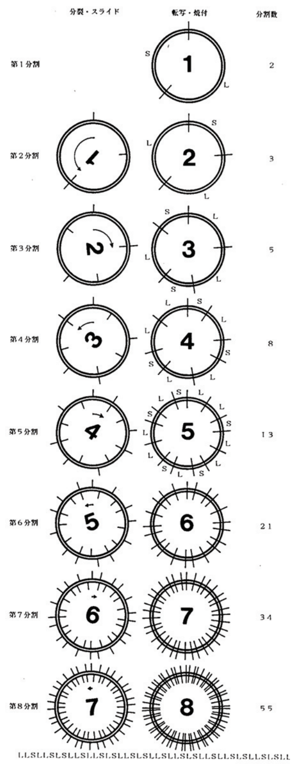

Fig. 5

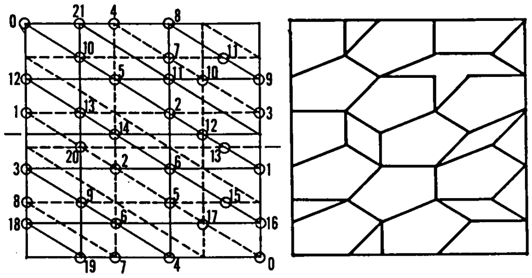

Fig. 6

As shown in Fig. 6, we can find the FIBONACCI LATTICE in Penrose Tiling.
It means that the FIBONACCI LATTICE is a one dimensional projection of the six dimensional lattice.

Expansion toward any real number

It was exciting to discover the quasi-periodic rhythm of the FIBONACCI LATTICE. It was enough to have an impact upon contemporary music. However, I expanded the principle to include all real numbers in 1992.
In the Tangent Method, when you set a line whose tangent is any value, you can get any rhythm of literal string repeated endlessly. So to speak, we can define a unique rhythm one by one corresponding to any real number. I call this idea "non-integer beat" [4].
When the tangent of a line, Q/P, is a rational number, the Tangent Method generates and repeats a literal string which consists of (P+Q) letters. During one period, the letter "A" appears "P" times, and the letter "B" appears "Q" times.
When the tangent of a line is an irrational number "R", the literal string must be non-periodic. The limit of convergence of the ratio of "B" to "A" is just the irrational number [5].
When the tangent of a line is set as the inverse number "1/R", "A" and "B" must also be inverted. Musicians call this a reverse rhythm.
In the case that the tangent is 1/1, there is no distinction between obverse and reverse, producing a duple beat.

Substitution Method	Non-Integer Beat (Tangent Method)
AB	1/1
ABA	1/2
ABAAB	2/3
ABAABABA	3/5
and so on..	

Fig. 7

From this viewpoint, after analysis about the relation between the Tangent and Substitution Method, I recognized a fact of the Fibonacci Lattice as shown in Fig. 7.
The Substitution Method generates a hierarchy of the string. Each string corresponds with each ratio of two successive numbers in the Fibonacci Lattice beautifully.
I composed a musical piece called the Fibonacci Kecak (1995) based on the above structure [6].

Not only the Golden Mean has such a hierarchy of strings and convergents; in fact, any real number has a unique structure.
I expanded the Fibonacci Kecak to work with any real number. This work was called the Real Kecak System (1999)[7].
One year after I published the Real Kecak System, I knew that the Euclidean Algorithm (continued fraction) determined the hierarchy of convergents [8], and then I updated the Real Kecak System including the algorithm [9].

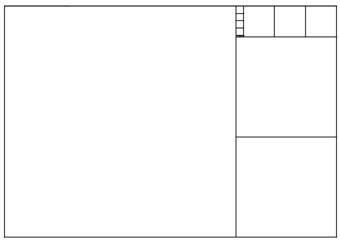

Fig. 8 Euclidean Algorithm

I understood the relation between the Tangent method and continued fraction clearly. But I could still not solve the relation between the Substitution Method and the continued fraction. In other words, how could I formulate the relation between the Tangent Method and Substitution Method?
The Substitution Method based on the Euclidean Algorithm can generate a non-periodic literal string. But it is not always identical with the Tangent one.
I solved this problem in the previous paper "Real Number Music" [1]. It was clarified during my experiment that I would apply the recursive structure of the continued fraction of any real number to musical scale and tone.

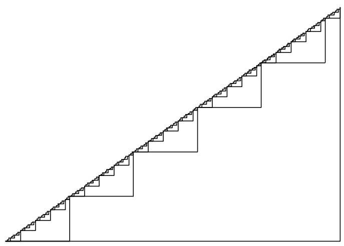

Fig. 9 Inverse-Euclidean Algorithm

Euclidean Algorithm and Inverse- Euclidean Algorithm

A real number generates two kinds of literal strings. I call them "REAL LATTICE".

Please refer to my previous paper "Real Number Music[1]. I already defined Euclidean Algorithm and Inverse-Euclidean Algorithm there. This paper provides a more detailed explanation.

Consider a real number "R" ($0 \leqq R \leqq 1$) which is expanded into the following regular continued fraction;

$$R = C_0 + \cfrac{1}{C_1 + \cfrac{1}{C_2 + \cfrac{1}{C_3 + \cfrac{1}{\ddots \, C_k + \cfrac{1}{\ddots}}}}}$$

Q_0/P_0	0^{th} convergent
Q_1/P_1	1^{st} convergent
Q_2/P_2	2^{nd} convergent
Q_3/P_3	3^{rd} convergent
Q_k/P_k	k^{th} convergent

Let $[C_0, C_1, C_2, C_3, \ldots, C_k, \ldots]$ denote the continued fraction. Since $0 \leqq R \leqq 1$, the "R" is always 0.

It is not essential whether the string is obverse or reverse because the letters "A" and "B" are completely exchangeable. Besides, the whole of real number can be mapped on the interval "$0 \leqq R \leqq 1$". Therefore I set the domain of definition of "R" as "$0 \leqq R \leqq 1$" so that I let the system be the simplest [10].

Let "R_k" denote the k-th convergent of "R", that is, $R_k = Q_k/P_k$ ($Q_k \leqq P_k$).

Consider the following string "X$", "Y$", and "Z$" which consist of letters "A" and "B".

When "X$" repeats "n" times, we denote it by "n*X$".

We denote a combination of "X$" and "Y$" as "X$+Y$".

On a literal string "X$", we denote an act that substitutes "Y$" into "A" and substitutes "Z$" into "B" as "X$(Y$, Z$)".

Let a string generated by Euclidean Algorithm be denoted by "R$".

Let a string generated by Inverse-Euclidean Algorithm be denoted by "R'$".

Let a string of "R$" and "R'$" corresponding with k-th convergent be denoted by "$R\$_k$" and "$R'\$_k$".

Each string will be generated the following process according to the partial denominator "C_k".

Euclidean Algorithm	Inverse-Euclidean Algorithm
$R\$_0 = A$	$X\$_0 = A$
$R\$_1 = R\$_0(C_1{*}A+B, A)$	$X\$_1 = X\$_0(C_k{*}A+B, A)$
$R\$_2 = R\$_1(C_2{*}A+B, A)$	$X\$_2 = X\$_1(C_{k-1}{*}A+B, A)$
$R\$_3 = R\$_2(C_3{*}A+B, A)$	$X\$_3 = X\$_2(C_{k-2}{*}A+B, A)$
:	:
$R\$_k = R\$_{k-1}(C_k{*}A+B, A)$	$R'\$_k = X\$_{k-1}(C_1{*}A+B, A)$

Note that I use "X$" as a parametric string on the Inverse-Euclidean Algorithm above. It is not always $X\$_k = R'\$_k$.

When a continued fraction contains a constant partial denominator as in the Golden Mean, it is always $R\$_k = R'\$_k$ and $X\$_k = R'\$_k$.

The number of times that letter "A" and "B" appear on the string is denominator "P_k" and numerator "Q_k" of the k-th convergent. It is identical with the one period of an infinite string which is generated by the Tangent Method and whose tangent is Q_k/P_k.

By the way, a rational number $[C_0, C_1, C_2, C_3 \ldots, C_k]$ which consists of a definite continued fraction has a dual rational number $[C_k, C_{k-1}, C_{k-2}, \ldots, C_3, C_2, C_1, C_0]$. The dual of rational number "R_k" is the ratio of the number of letters "B" to "A".

In particular, when a continued fraction contains a constant partial denominator, there is no difference between the Euclidean Algorithm and the Inverse-Euclidean Algorithm. We should call such type of rational number a "self-dual rational number."

A way to know the number of letters "A" and "B" on the Euclidean Algorithm

We can formulate how to know the number of letters "A" and "B" on the Euclidean Algorithm.

Let "PP_k" denote the number of letters "A" on the k-th String of the Euclidean Algorithm.

Let "QQ_k" denote the number of letters "B" on the k-th String of the Euclidean Algorithm. Note that the "QQ_k" is always PP_{k-1}.

Set first terms: $PP_{-1}=0$, $QQ_{-2}=1$.

Recurrence formula: $PP_k=C_k*PP_{k-1}+PP_{k-2}$

For example, examine a rational number [0,3,1,2] which is asymmetric arrangement of partial denominator.

k	R	continued fraction	convergent QK/PK	R$k	R'$k
0	3/11	= 0+ 1/	0/1	A	A
1		3+ 1/	1/3	AAAB	AAAB
2		1+ 1/	1/4	ABABABA	AAABA
3		2	3/11	AABAAABAAABAAAB	AAABAAAABAAAAB
				QQ_3/PP_3=4/11 dual	3/11

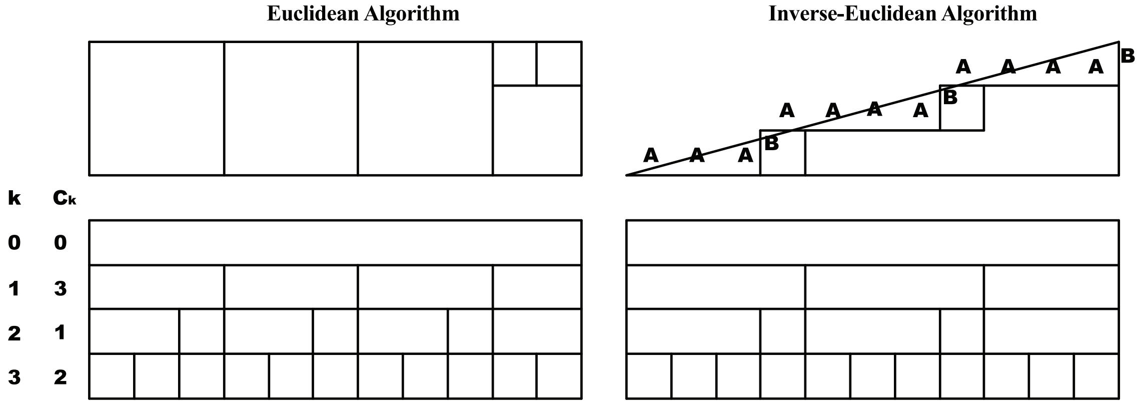

Fig. 10 In case R=3/11

Fig. 11

R'$-2 = A
R'$-1 = B
R'$0 = C0*R'$-1+ R'$-2
R'$1 = C1*R'$0 + R'$-1
R'$2 = C2*R'$1 + R'$0
R'$3 = C3*R'$2 + R'$1
:
R'$k = Ck*R'$k-1+ R'$k-2
:

Formula of the Substitution Method

In general, when a real number "R" is expanded into a continued fraction as [$C_0,C_1,C_2,C_3,\ldots,C_k,\ldots$], its literal string of the k-th Inverse-Euclidean Algorithm "R'$k" will be generated as shown on Fig. 11.

Since "C_0" is always "0", "R'$0" is always "A". However, it is acceptable to have 1<R in this formula on left side.

Conclusion

There are two kinds of strings that are generated by the Substitution Method.

One is the Euclidean Algorithm. Another is the Inverse-Euclidean Algorithm.

The Inverse-Euclidean Algorithm is equivalent to the Tangent Method.

The Inverse-Euclidean Algorithm possesses a self-similar structure based on the recursive hierarchy of continued fractions, as I showed in the previous paper "Real Number Music".

The Euclidean Algorithm is equivalent to the method shown in Figs. 2, 3, 4, and 5.

Real Lattice generated by Chaotic Dynamic System

I found a kind of Chaotic Dynamic System[11] which also generates the same string as the Real Lattice as shown in Fig. 12.
Start from any point and follow the arrows along the locus.
Your history traveling between A and B area is the same string as the Inverse-Euclidean Algorithm.
Incidentally, the pattern which divides the oblique line e-f is the same as the Euclidean Algorithm.

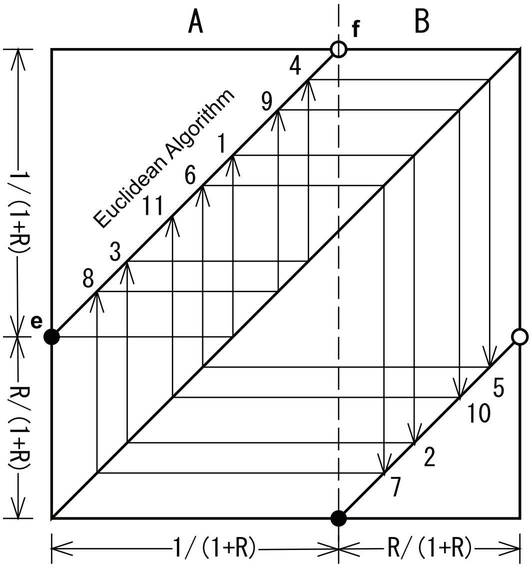

Fig. 12 General Real Lattice

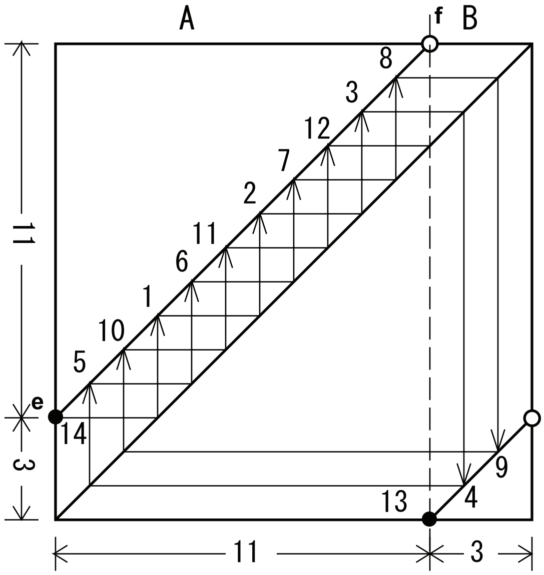

Fig. 13 In case R=3/11

Acknowledgement

I would like to thank Mr. Peter Engel who gave me suggestions for my English expression.

References

[0]This paper is a translation of the Japanese one which was published in MANIFOLD #04, 2002.
[1]Akio HIZUME “Real Number Music,” MANIFOLD #03, 2001.
[2]Toru OGAWA “3D Penrose Conversion,” Sugaku Seminer 1986.
This quasi-periodic string was found by Fibonacci 800 years ago.
[3]Akio HIZUME “Life and Architecture” 1990. Figs. 2, 3, 4, 5, and 6 are reprints from the book.
[4]Akio HIZUME “Quasi-Periodic Electric Instrument,” Japan Patent Office.
Akio HIZUME “Golden Music,” KATACHI-NO-BUNKA-SHI, KOSAKU-SHA, 1994.
[5]There is another choice of definition in which “P” is the total number of the letters and “Q” is the number of letter “A.” But it is too complicated to include in the body of this paper.
[6]I published the musical work in ISIS SYMMETRY, Aug. 1995, Washington D.C.
[7]I wrote the original program in N88 Basic first.
[8]Prof. Kazuo AZUKAWA encouraged my interest in the continued fraction.
[9]Akio HIZUME “Real Kecak System,” MANIFOLD #02, 2001.
[10] There is another choice that the “R” has no domain definition. In that case, it is too complicated to include in the body of this paper.
[11]Masaya YAMAGUCHI “Chaos and Fractal,” KODAN-SHA 1986.